<u>Course info:</u>

- Denis Auroux, 817 Evans, auroux@math
  office hrs: generally after class / by appointment
- This continues 215A (= fundamental group, covering spaces, homology, cohomology)

  Main topics:
  - homotopy groups $\pi_n(X)$ & relation to H_n, H^n
  - fibrations & their homotopy / (co)homology
    ↳ spectral sequences
  - characteristic classes of vector bundles
- Books: Hatcher chapter 4 (available online)
  Milnor-Stasheff "Charact. classes"
  lecture notes by Hutchings
- Homework every 2-3 weeks.
- NEXT WEEK no class Wed 1/25, Fri 1/27 - will make up during RRR week.

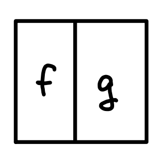
<u>Homotopy groups:</u> $\pi_n(X)$ generalizes π_1 = homotopy classes of based loops $(S^1,*)\to(X,x_0)$ to homotopy classes of based maps $(S^n,*)\to(X,x_0)$

At first glance may seem similar to H_n, and indeed closely related, but still quite different... e.g. $H_*(S^n)$ easy to compute, $\pi_*(S^n)$ very hard (& still open problem!!)

- <u>Definition:</u> Let $I^n=[0,1]^n$ unit cube, X space with base point x_0,
  $\pi_n(X,x_0)=$ homotopy classes of maps $f:(I^n,\partial I^n)\longrightarrow(X,x_0)$.
  (where homotopies should satisfy $f_t(\partial I^n)=x_0\ \forall t$).

  <u>Remarks:</u>
  - for $n=1$, agrees with π_1. (intervals w/ both ends at $x_0\Leftrightarrow$ loops)
  - for $n=0$, by convention I^0 = point, $\partial I^0=\emptyset$,
    $\pi_0(X,x_0)=\{\text{path conn. components of } X\}$

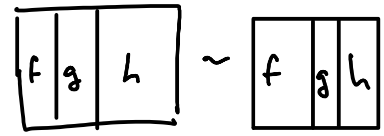
- For $n\geq 2$, define a sum operation on π_n (extends product on π_1):

<u>Def.</u>
- $(f+g)(s_1,\dots s_n)=\begin{cases} f(2s_1,s_2\dots s_n) & s_1\leq\frac{1}{2}\\ g(2s_1-1,s_2\dots s_n) & s_1\geq\frac{1}{2}\end{cases}$ ie. 

- $(-f)(s_1\dots s_n)=f(1-s_1,s_2\dots s_n)$

agrees with def. on π_1.

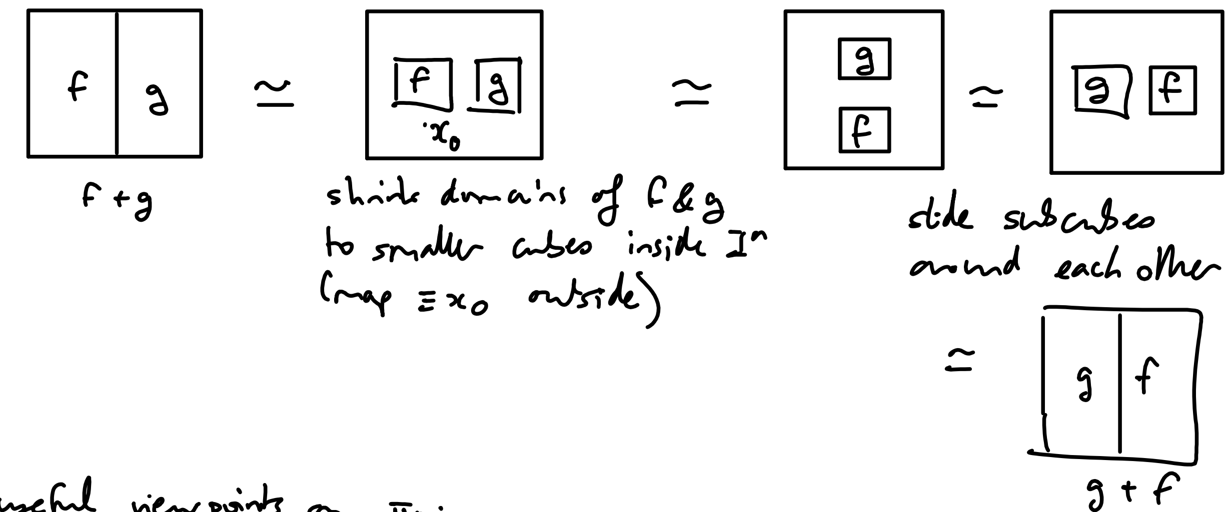
defines a group (identity = constant map $I^n \to x_0$)
(associativity up to homotopy: same as for π_1



also $f + (-f) \sim id$ ✓)

- However: **for $n \geq 2$, $+$ is commutative & $\pi_n(X, x_0)$ is an abelian group**

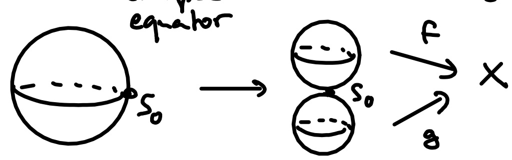
PF:



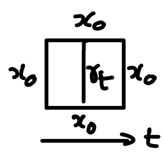
Two other useful viewpoints on π_n:

1) • $I^n/\partial I^n \simeq D^n/\partial D^n \simeq S^n$, so maps $(I^n, \partial I^n) \to (X, x_0)$
are the same as maps $(I^n/\partial I^n, \partial I^n/\partial I^n) \to (X, x_0)$
hence $(S^n, s_0) \longrightarrow (X, x_0)$.

• addition is then: $S^n \xrightarrow[\text{collapse equator}]{c} S^n \vee S^n \xrightarrow[f \vee g]{} X$

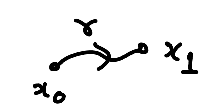


2) • Can think of π_2 as loops of based loops in X ($\gamma_t(s) = f(t,s)$)
So $\pi_2(X) \simeq \pi_1(\Omega X)$



loopspace ΩX = based loops in (X, x_0) = $\{\text{maps } (I, \partial I) \to (X, x_0)\}$, w/ base pt = const loop
compact-open topology $\mathcal{U}_{K,U} = \{f \mid f(K) \subset U\}$ ($\leftrightarrow$ local unif. cv topology)
By def. of sum operation, this is a group isomorphism.
Similarly, $\pi_n(X) \simeq \pi_{n-1}(\Omega X)$. $(\gamma_{(t_1, \dots t_{n-1})}(s) = f(t_1, \dots t_{n-1}, s))$

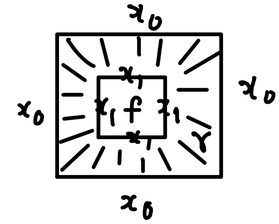
Dependence on base point: assume X path connected; $\gamma : I \to X$ path



**Induces an isomorphism $\gamma_\# : \pi_n(X, x_1) \xrightarrow{\sim} \pi_n(X, x_0)$
with $\gamma_\# \cdot \eta_\# \simeq (\gamma \cdot \eta)_\#$ and $(\gamma^{-1})_\# = (\gamma_\#)^{-1}$**

Proof: given $f: (I^n, \partial I^n) \to (X, x_1)$, define $\gamma \cdot f =$

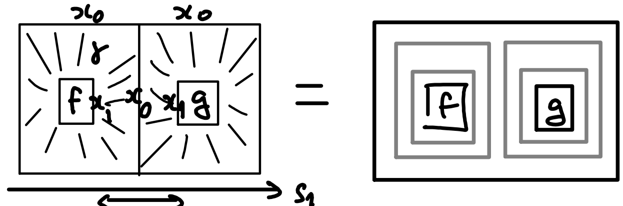
(f in subcube,
interpolate $x_0 \to x_1$ radially along γ).



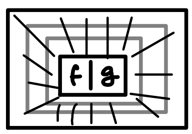
ie. if $\tau = \sup_{1 \le i \le n} \left(|t_i - \frac{1}{2}|\right) \ge \frac{1}{4}$,
take $\gamma(4(\frac{1}{2} - \tau))$

clearly $(\gamma\eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$

claim $(\gamma \cdot f) + (\gamma \cdot g) \simeq \gamma \cdot (f + g)$



squish this part (use symmetry: skip from $s_1 = \frac{1}{2} - \varepsilon(t)$ to $\frac{1}{2} + \varepsilon(t)$.

$\xrightarrow{\text{homotopy}}$  $\simeq \gamma \cdot (f+g)$.

So $[f] \mapsto [\gamma \cdot f]$ (clearly well-def^d) is a group homomorphism $\gamma_\#$.

Clearly $\gamma_\# \cdot \eta_\# = (\gamma\eta)_\#$; $\gamma_\#$ is an isom. because $(\gamma^{-1})_\# \gamma_\# = id$. ☐

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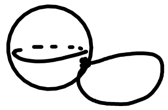
So can write $\pi_n(X)$ & forget base pt ... but not too much! as in the case of π_1, these isos are noncanonical! in fact, considering case where γ = loop in (X, x_0), get an <u>action of $\pi_1(X, x_0)$</u> on $\pi_n(X, x_0)$ by automorphisms!

(for $n = 1$, $\gamma_\#([f]) = [\gamma f \gamma^{-1}]$ is conjugation by $[\gamma]$ in π_1).

This make $\pi_n(X)$ a $\mathbb{Z}[\pi_1(X)]$-module for $n \ge 2$.
(Even though π_n is abelian, it can still have a nontrivial module structure over π_1!)

(<u>Exercise</u> (<u>HW</u>):  $\pi_2(S^2 \vee S^1)$? module structure ?)

<u>Basic properties of π_n:</u>

- π_n is a <u>functor</u>, ie. a map $\varphi: (X, x_0) \to (Y, y_0)$ induces $\varphi_*: \pi_n(X, x_0) \to \pi_n(Y, y_0)$
$$\varphi_*[f] = [\varphi \circ f].$$

clearly, $\varphi \underset{\text{homotopic}}{\simeq} \varphi' \Rightarrow \varphi_* = \varphi'_*$; $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$.

In particular a homotopy equivalence induces isoms. on π_n.

- Covering spaces:

Prop: $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering $\Rightarrow \forall n \geq 2$, $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism

Pf: every map $(S^n, s_0) \to (X, x_0)$ lifts to $\tilde{X}$ for $n \geq 2$ (S^n simply connected) (uniquely). This yields a map $\pi_n(X, x_0) \to \pi_n(\tilde{X}, \tilde{x}_0)$, inverse to p_*. $\blacktriangle$

In particular, if the universal cover $\tilde{X}$ of X is contractible then $\pi_n(X) = 0$ $\forall n \geq 2$!

Eg: for the torus $T^n = (S^1)^n$, the univ cover is $\mathbb{R}^n$, so $\pi_i(T^n) = 0$ $\forall i \geq 2$. (say T^n **aspherical**).
(very different from homology $H_i(T^n) \neq 0$ $\forall i \leq n$).

- Products:

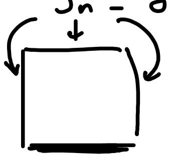
Prop: $\pi_n\left(\prod_{i \in I} X_i\right) = \prod_{i \in I} \pi_n(X_i)$

Pf: maps $S^n \xrightarrow{f} \prod_i X_i \iff$ collection of maps $f_i : S^n \to X_i$ $\forall i$
same for homotopy $S^n \times I \to \prod_i X_i \iff$ collection of homotopies. $\blacktriangle$

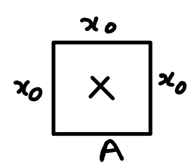
Again much simpler than homology (Künneth formula).

__Relative homotopy groups:__ for a pair (X, A) + base point $x_0 \in A$:

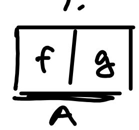
$J_n = \partial I^n \setminus I^{n-1}$

$I^n =$ 

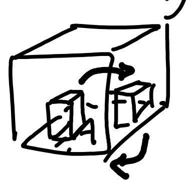
$I^{n-1} = I^{n-1} \times \{0\}$

$\Rightarrow$ for $n \geq 1$,
$\pi_n(X, A, x_0) =$ homotopy classes of maps $(I^n, \partial I^n, J_n) \to (X, A, x_0)$
ie. 

Alternatively, since $(I^n, \partial I^n)/J_n \simeq (D^n, S^{n-1})$, can think of
$\pi_n(X, A, x_0) =$ homotopy classes of maps $(D^n, \partial D^n = S^{n-1}, s_0) \to (X, A, x_0)$
(so eg $\pi_2(X, A) =$ discs in X w/ boundary in A).

Addition = as before, concatenate in 1st coordinate .

Commutation trick only works for $n \geq 3$ though (can't use n^{th} coordinate as before)

can do  so: $\pi_n(X, A, x_0) = \begin{cases} \text{set for } n = 1 \\ \text{group for } n = 2 \\ \text{abelian gp for } n \geq 3. \end{cases}$

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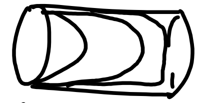
- <u>Compression criterion:</u> $f:(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents 0 in $\pi_n(X,A,x_0)$ iff it is homotopic rel S^{n-1} to a map with image $\subseteq A$

<u>Pf:</u>
- if $f \underset{\text{rel }\partial}{\simeq} g$ with $g(D^n) \subseteq A$ then $[f]=[g]$ and $[g]=0$ since can compose g with deformation retraction of D^n to s_0 to get $g \simeq$ const map.

- conversely, if $f \simeq 0$ via homotopy $F: D^n \times [0,1] \to X$ then compose F with homotopy



$D^n \times \{0\} \underset{\text{rel }\partial}{\rightsquigarrow} S^{n-1} \times [0,1] \cup D^n \times \{1\}$

to get a homotopy from f to map w/ values in A, stationary on S^{n-1}.

- $\varphi: (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms on π_n, as before.

- <u>Long exact sequence:</u> (cf. relative homology!)

<u>Thm:</u> $\exists$ long exact sequence

$$\cdots \to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \xrightarrow{i_*} \cdots \to \pi_0(X,x_0)$$

$i: A \hookrightarrow X$ inclusion
$j: (X,x_0) \to (X,A)$

∂: restriction to boundary (from I^n to I^{n-1} or D^n to S^{n-1}).

(& similarly for a triple (X,A,B) with $x_0 \in B \subset A \subset X$, relative π_n's).

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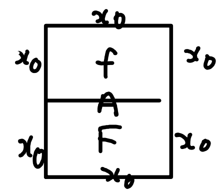
<u>Pf:</u>
- exactness at $\pi_n(X,x_0)$: $j_* \circ i_* = 0$ by compression criterion (maps to A represent zero in $\pi_n(X,A)$).

Assume $f: (I^n, \partial I^n) \to (X, x_0)$ represents zero in $\pi_n(X,A)$, then by compression criterion can homotope f (keeping $\partial I^n \to x_0$) to a map $(I^n, \partial I^n) \to (A, x_0)$ thus $[f] \in \operatorname{Im} i_*$.

- exactness at $\pi_n(X,A)$: $\partial \circ j_* = 0$ since boundary restriction of a map $(I^n, \partial I^n) \to (X, x_0)$ is constant; conversely, assume $f: (I^n, \partial I^n, J_n) \to (X, A, x_0)$ has $\partial[f] = 0$, ie. $f_{|I^{n-1}} \simeq$ constant, by homotopy $F: I^{n-1} \times [0,1] \to A$. Then f is homotopic among rel. $\partial I^{n-1} \to x_0$

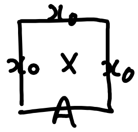
maps $(I^n, \partial I^n, J) \to (X, x_0)$ to

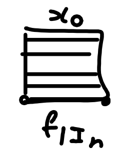


$= \tilde{f}: (I^n, \partial I^n) \to (X, x_0)$

(stack f with truncations of F) so $[f] \in \operatorname{Im} j_*$.

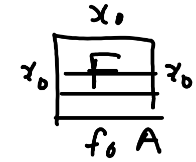
- exactness at $\pi_n(A,x_0)$: $i_* \circ \partial = 0$ since restriction to I^n of

$f =$  is homotopic rel. ∂I^n to constant map at x_0 through f itself!



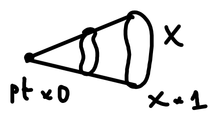
$f: (I^{n+1}, \partial I^{n+1}, J_{n+1}) \to (X, A, x_0)$

Conversely: if $i_*([f_0]) = 0$ then $f_0: (I^n, \partial I^n) \to (A, x_0)$ is homotopic rel. ∂I^n to constant map in X

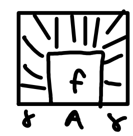


The homotopy gives $F: (I^{n+1}, \partial I^{n+1}, J_{n+1}) \to (X, A, x_0)$, $\partial([F]) = [f_0]$. ▲

__Ex.:__ Cone $CX = X \times I / X \times \{0\}$ (X path connected)



CX contractible $\Rightarrow$ get
$$\underbrace{\pi_n(CX)}_{=0} \to \pi_n(CX, X) \xrightarrow[\text{isom.}]{\simeq} \pi_{n-1}(X) \to \underbrace{\pi_{n-1}(CX)}_{=0}$$

The l.e.s. is natural wrt maps of pairs; and changes of basepoints induce isoms on relative π_n's ($\gamma \cdot f =$ )

__Def.:__ (X, x_0) is __n-connected__ if $\pi_k(X, x_0) = 0$ $\forall k \le n$.

0-connected = path conn. ($\Rightarrow$ x_0 doesn't matter)
1-connected = simply conn.

$\pi_k(X, x_0) = 0$ $\forall x_0 \in X$ $\Leftrightarrow$ every map $S^k \to X$ is homotopic to a constant map
$\Leftrightarrow$ ——— " ——— extends to a map $D^{k+1} \to X$

Similarly for pairs,

$\pi_k(X, A, x_0) = 0$ $\forall x_0 \in A$

$\Leftrightarrow$ every map $(D^k, \partial D^k) \to (X, A)$ is homotopic rel. ∂D^k to a map $D^k \to A$
$\Leftrightarrow$ ——— " ——— among such maps to a map $D^k \to A$
$\Leftrightarrow$ ——— " ——— to a constant map.

Say (X, A) is n-connected if holds $\forall k \le n$.